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Further Modification of Bolotin Method in Vibration Analysis of Rectangular Plates

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A further modification of the Bolotin method for the determination of the natural frequencies and mode shapes of isotropic and orthotropic rectangular plates with various types of boundary conditions is given. Unlike the Bolotin method (BM) or the modified Bolotin method (MBM), the present approach does not postulate the formula for the eigenfrequency, but rather is based on the condition that the frequency obtained from the governing differential equations has to be equal to that yielded by Rayleigh's method. This modification is shown to be more straightforward and faster in computation, and the mode shapes derived are valid on a larger portion of the plate. Furthermore, the proposed modification easily provides a solution for boundary conditions for which the BM and MBM cannot provide a solution. Problems with two different sets of boundary conditions were solved in this study: a rectangular orthotropic plate with all edges clamped and rectangular isotropic and orthotropic plates clamped along one pair of opposite edges and free along the other pair. The results obtained for the first set compared favorably with those yielded by the MBM and Rayleigh methods, whereas in the second case the BM and MBM failed to predict the beam-like modes of vibration, while the present modification treats the problem satisfactorily.

Nomenclature

$A_1 \ldots A_4$	= constants
a, b	= lengths of the side of the plate along the x and y axes, respectively
c_1, c_2, C_1, C_2	= real numbers
D_x	= flexural rigidity in x direction, $D_x = E_x h^3/12(1 - \nu_x \nu_y)$
D_{xy}	= flexural rigidity from the coupling of x and y directions, $D_{xy} = D_x \nu_y$
D_y	= flexural rigidity in y direction, $D_y = E_y h^3/12(1 - \nu_x \nu_y)$
D_{66}	= torsional flexural rigidity, $D_{66} = G h^3/12$
E_x, E_y	= Young's moduli along the x and y axes, respectively
G	= rigidity modulus
H	= $D_{xy} + 2D_{66}$
h	= plate thickness
K	= kinetic vibration energy
m, n	= integer numbers
p, q	= real numbers
s_1, s_2, S_1, S_2	= real numbers
t	= time
w	= transverse displacement
X	= mode shape function in x direction
x, y	= rectangular coordinates
Y	= mode shape function in y direction
θ	= $[2q^2 H/D_x + p^2]^{1/2}$
κ	= $[2p^2 H/D_y + q^2]^{1/2}$
$\lambda, \lambda^*, \tilde{\lambda}$	= real or imaginary numbers, roots of characteristic equation
ν_x, ν_y	= Poisson's ratios for the material where $E_x \nu_y = E_y \nu_x$
Π	= potential vibration energy
ρ	= mass density
τ	= direction normal to the contour of the plate
ω	= eigenfrequency or cycle frequency

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I. Introduction

THE concept of the dynamic-edge-effect method known as the Bolotin method (BM) was first introduced for solving plate and shell problems in 1960.[1,2] In this method eigenmodes are initially approximated by a general solution, consisting of trigonometric functions, in the interior region of the prescribed boundaries of a plate or shell element. Suitable correction terms are then added in order to determine mode shapes for the given boundary conditions.[1–3] This method attracted much interest in the former Soviet Union. Many problems were solved by this method, including frequency and mode shapes of plates under various edge conditions, free vibration and stability of rectangular, spherical, conical, and other types of shells, as well as certain simple cases that make use of refined plate theories (for instance, those of Reissner, Mindlin, and Ambartsumyan). About 100 investigations performed in the former Soviet Union in which the BM was applied were briefly reviewed in Ref. 4.

More recently the BM began to receive wide attention in the West also. One of these early Western studies[5] points out that the main advantage of the method is that it yields great accuracy for high modes with no more computational effort than required by conventional methods such as finite elements, finite differences, Rayleigh–Ritz and Fourier series. Conventional methods provide accurate results only for low modes. The study also indicates that when using the BM for a plate clamped along one pair of opposite edges and free along the other pair the modes corresponding to beam-like vibrations are missed altogether. (It will be shown that this restriction has been overcome in the present proposed modification of the BM). In another investigation[6] the BM was applied in studying the vibration of multiplate systems and box-type structures. In Ref. 7 the BM was applied to vibration and buckling of rectangular orthotropic plates subjected to uniform in-plane loads, but a comment[8] pointed out that here the BM failed to predict the buckling load for certain combinations of the elasticity parameters.

Concurrently with these recent developments, the modified Bolotin method (MBM) was introduced.[3,9] This approach improves the BM by retaining nondecaying terms, which were neglected by Bolotin. The MBM led to very accurate results, especially in calculations of high modes, and was found to be more convenient to apply than the BM. Furthermore, some problems that the BM failed to treat were solvable by the MBM. With the MBM the problems of buckling and vibration of in-plane loaded rectangular orthotropic plates could be solved for any elasticity ratios.[10,11] In recent studies[12,13] the BM or the MBM were used in combination with other methods. In Ref. 12 Bolotin asymptotic solutions were used as admissible functions in the application of the Rayleigh–Ritz method. Infinite series of Bolotin functions were introduced into the solution of the vibration equation to obtain the natural frequencies of stiffened plates and of plates with elastically attached masses in Ref. 13.

As indicated in Ref. 3, both the eigenvalues and the eigenvectors can be obtained by application of the MBM. However, it was shown in Ref. 12 that when these eigenmodes were used with Rayleigh's energy method slightly higher frequencies were calculated. The present study aims at showing that the use of the present proposed modification of the BM avoids this discrepancy. In addition the study will demonstrate that this modification leads to higher accuracy, to solving the problems that are not solvable by BM or MBM, and at the same time apparently can reduce the calculation time significantly. Although the present approach is based on thin-plate theory, it is nevertheless equally applicable when considering shear deformation theory in the case of thicker plates.

All computations in this investigation were performed using the Mathematica program.[14]

II. Analytical Formulation

A description of the MBM is given in Ref. 3. There, the problem of free small amplitude flexural vibration of a thin orthotropic plate is described by the classical differential equation

$$D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2\, \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} - \rho h \omega^2 w = 0 \tag{1}$$

In this study the proposed modified method is used to solve Eq. (1) for two sets of boundary conditions. The first case is associated with a plate with all sides clamped, for which the boundary conditions are given by

$$w = \frac{\partial w}{\partial \tau} = 0 \tag{2}$$

In the second case a square plate clamped along one pair of its opposite edges and free along its other pair of edges is treated. Here, for the clamped edges $x = 0$ and a (where a is the length of square plate), Eq. (2) applies. Along the free edges $y = 0$ and a, the following relations have to be fulfilled[15]:

$$D_{xy} \frac{\partial^2 w}{\partial x^2} + D_y \frac{\partial^2 w}{\partial y^2} = 0, \qquad H \frac{\partial^3 w}{\partial x^2 \partial y} + D_y \frac{\partial^3 w}{\partial y^3} = 0 \tag{3}$$

For the sake of completeness, a condensed description of the MBM in Ref. 3 is given here. A description of the application of the proposed modification for the plate with all edges clamped is presented next. All that is needed to derive the results for any other sets of boundary conditions is to replace Eq. (2) in compliance with the new boundaries.

A. MBM

In the MBM an eigenmode is initially approximated by a general solution consisting of trigonometric functions (in the interior region):

$$w_{mn} = A_{mn} \cdot \sin[(p\pi/a)(x - x_0)] \cdot \sin[(q\pi/b)(y - y_0)] \tag{4}$$

p, q are real numbers such that $m \le p \le m + 1$ and $n \le q \le n + 1$ and are to be determined.

From Eqs. (4) and (1) the formula for the eigenfrequency derived by the MBM is given by

$$\omega = \left(\frac{H}{\rho \cdot h}\right)^{\frac{1}{2}} \cdot \left[\frac{D_x}{H}\left(\frac{p\pi}{a}\right)^4 + 2\left(\frac{p\pi}{a}\right)^2\left(\frac{q\pi}{a}\right)^2 + \frac{D_y}{H}\left(\frac{q\pi}{a}\right)^4\right]^{\frac{1}{2}} \tag{5}$$

Then, the MBM considers two auxiliary Levy-type problems.

First problem. The class of solution of Eq. (1) is assumed as

$$w_{mn} = Y(y) \cdot \sin[(p\pi/a)(x - x_0)] \tag{6}$$

satisfying the boundary conditions

$$Y = \frac{\mathrm{d}Y}{\mathrm{d}y} = 0 \qquad \text{at} \qquad y = 0,\ a \tag{7}$$

Substituting Eq. (6) into Eq. (1) with ω from Eq. (5) results in an ordinary differential equation for $Y(y)$:

$$\frac{\mathrm{d}^4 Y}{\mathrm{d}y^4} - 2\frac{H}{D_y}\left(\frac{p\pi}{a}\right)^2 \frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} - \left(\frac{q\pi}{a}\right)^2\left[2\frac{H}{D_y}\left(\frac{p\pi}{a}\right)^2 + \left(\frac{q\pi}{a}\right)^2\right] Y = 0 \tag{8}$$

for which the corresponding characteristic equation is given by

$$\lambda^4 - 2\frac{H}{D_y}\left(\frac{p\pi}{a}\right)^2 \lambda^2 - \left(\frac{q\pi}{a}\right)^2\left[2\frac{H}{D_y}\left(\frac{p\pi}{a}\right)^2 + \left(\frac{q\pi}{a}\right)^2\right] = 0 \tag{9}$$

This characteristic equation possesses two imaginary and two real roots; namely,

$$\lambda_{1,2} = \pm i(q\pi/a), \qquad \lambda_{3,4} = \pm(\pi/a)\kappa \tag{10}$$

where

$$\kappa = \left[2p^2 H \big/ D_y + q^2\right]^{\frac{1}{2}}$$

The general solution of Eq. (8) can be represented as

$$Y(y) = A_1 \cosh\left(y\frac{\pi\kappa}{a}\right) + A_2 \sinh\left(y\frac{\pi\kappa}{a}\right) + A_3 \cos\left(y\frac{\pi q}{a}\right) + A_4 \sin\left(y\frac{\pi q}{a}\right) \tag{11}$$

Substitution of Eq. (11) into the boundary condition, Eq. (2), and the requirement for the existence of a nontrivial solution yield the characteristic determinant

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \kappa & 0 & q \\ C_2 & S_2 & c_2 & s_2 \\ \kappa S_2 & \kappa C_2 & -q s_2 & q c_2 \end{bmatrix} = 0 \tag{12}$$

where

$$c_2 = \cos(q\pi), \qquad s_2 = \sin(q\pi)$$
$$C_2 = \cosh(\kappa\pi), \qquad S_2 = \sinh(\kappa\pi) \tag{13}$$

Expanding Eq. (12) leads to

$$1 - c_2 C_2 + \frac{H}{D_y}\left(2\frac{H}{D_y}\frac{q^2}{p^2} + \frac{q^4}{p^4}\right)^{-\frac{1}{2}} s_2 S_2 = 0 \tag{14}$$

Second problem. Now another class of solutions is considered:

$$w_{mn} = X(x) \cdot \sin[(q\pi/a)(y - y_0)] \tag{15}$$

which satisfy the boundary conditions

$$X = \frac{\mathrm{d}X}{\mathrm{d}x} = 0 \qquad \text{at} \qquad x = 0, a \tag{16}$$

In a way entirely analogous to the derivation of Eq. (14), the following characteristic equation is obtained:

$$1 - c_1 C_1 + \frac{H}{D_x}\left(2\frac{H}{D_x}\frac{p^2}{q^2} + \frac{p^4}{q^4}\right)^{-\frac{1}{2}} s_1 S_1 = 0 \tag{17}$$

where

$$c_1 = \cos(p\pi), \qquad s_1 = \sin(p\pi), \qquad C_1 = \cosh(\theta\pi)$$

$$S_1 = \sinh(\theta\pi), \qquad \theta = \left[2q^2 H \big/ D_x + p^2\right]^{\frac{1}{2}} \tag{18}$$

The unknown quantities p and q are calculated by solving the system of transcendental Eqs. (14) and (17). By substituting p and q into the postulated Eq. (5), MBM leads to the natural frequency. With p and q evaluated from Eq. (12), all of the constants of integration of Eq. (11) $A_1 \ldots A_4$ can be determined, and the function $Y(y)$ can be found. In a analogous manner the function $X(x)$ can be found as well:

$$\begin{aligned} X(x) &= \cosh\left(x\frac{\theta\pi}{a}\right) - \cos\left(x\frac{p\pi}{a}\right) \\ &\quad - \left(\frac{S_1 - \theta p^{-1} s_1}{C_1 - c_1}\right)^{-1}\left[\sinh\left(x\frac{\theta\pi}{a}\right) - \frac{\theta}{p}\sin\left(x\frac{p\pi}{a}\right)\right] \\ Y(y) &= \cosh\left(y\frac{\kappa\pi}{a}\right) - \cos\left(y\frac{q\pi}{a}\right) \\ &\quad - \left(\frac{S_2 - \kappa q^{-1} s_2}{C_2 - c_2}\right)^{-1}\left[\sinh\left(y\frac{\kappa\pi}{a}\right) - \frac{\kappa}{q}\sin\left(y\frac{q\pi}{a}\right)\right] \end{aligned} \tag{19}$$

The mode shapes are determined as the product

$$w_{mn} = X(x) \cdot Y(y) \tag{20}$$

The derivation of the eigenfrequencies and eigenmodes by the MBM was described in Ref. 3. Now, the eigenmodes are known, and the frequency of any mode can be derived by Rayleigh's energy method. The potential vibration energy of a clamped thin plate in a given mode is described by

$$\begin{aligned} \Pi(w_{mn}) = \int_0^a\int_0^a \frac{1}{2}\Bigg[& D_x\left(\frac{\partial^2 w_{mn}}{\partial x^2}\right)^2 + 2D_{xy}\frac{\partial^2 w_{mn}}{\partial x^2}\frac{\partial^2 w_{mn}}{\partial y^2} \\ & + D_y\left(\frac{\partial^2 w_{mn}}{\partial y^2}\right)^2 + 4D_{66}\left(\frac{\partial^2 w_{mn}}{\partial x \partial y}\right)^2\Bigg]\mathrm{d}x\,\mathrm{d}y \end{aligned} \tag{21}$$

The kinetic energy corresponding to this mode is given by

$$K(w_{mn}) = \int_0^a\int_0^a \frac{1}{2}\rho h w_{mn}^2\,\mathrm{d}x\,\mathrm{d}y \tag{22}$$

Estimates of eigenfrequencies by Rayleigh's method are obtained from

$$\omega_{mn} = \sqrt{\frac{\Pi(w_{mn})}{K(w_{mn})}} \tag{23}$$

This frequency is larger than the frequency of the same mode obtained by the MBM. The percentage difference between Rayleigh's and MBM estimates decrease with increasing value of $\min(m, n)$, but it persists even at rather high modes (Table 1).

B. Description of the Present Approach

For calculating the frequency w_{mn} of any mode, it is initially assumed that, as in Eq. (5),

$$\omega_{mn} = \left(\frac{H}{\rho\cdot h}\right)^{\frac{1}{2}} \cdot \left[\frac{D_x}{H}\left(\frac{p_1\pi}{a}\right)^4 + 2\left(\frac{p_1\pi}{a}\right)^2\left(\frac{q_1\pi}{a}\right)^2 + \frac{D_y}{H}\left(\frac{q_1\pi}{a}\right)^4\right]^{\frac{1}{2}} \tag{24}$$

Table 1 Estimates of the frequency parameter $\bar{\omega} = (a/\pi)^2\omega(\rho h/H)^{1/2}$[a]

Number	n	m	MBM	Present method	Rayleigh method
1	1	1	3.58986	3.67686	3.67622
2	1	2	6.70818	6.77324	6.77298
3	2	1	8.11452	8.15463	8.15453
4	2	2	10.95921	11.03524	11.03496
5	1	3	11.68761	11.72522	11.72517
6	3	1	15.00631	15.02706	15.02704
7	2	3	15.72654	15.79938	15.79919
8	3	2	17.74384	17.79906	17.79900
9	1	4	18.40220	18.42537	18.42536
10	2	4	22.31569	22.37395	22.37387
11	3	3	22.35201	22.42450	22.42449
12	4	1	24.21112	24.22350	24.22348
13	4	2	26.90387	26.94226	26.94222
14	3	4	28.80016	28.87300	28.87289
15	4	3	31.42266	31.48289	31.48283
16	4	4	37.76454	37.83538	37.83530
17	8	10	171.21867	171.28794	171.28792
18	10	8	183.14374	183.20459	183.20458
19	0	10	191.76379	191.83293	191.83291
20	10	9	198.03292	198.09774	198.09772
21	10	10	214.64844	214.72006	214.72004
22	20	20	831.03480	831.10123	831.10123

[a] $D_x = 35.2026$ Nm; $D_{xy} = 14.5893$ Nm; $a = 0.1$ m; $D_y = 65.4616$ Nm; $D_{66} = 17.3863$ Nm; $h = 0.00216$ m; $\rho = 1650$ kg/m^3.

where $p_1 = m + 0.5$ and $q_1 = n + 0.5$. This permits the start of the iteration process. Again, as in the preceding case, two Levy-type problems have to be solved.

First problem. As before, the class of solution of Eq. (1) is assumed to be as in Eq. (6). Equation (6) is substituted into the governing Eq. (1), where ω from Eq. (24) is used. The following Levy-type problem is derived:

$$\frac{\mathrm{d}^4 Y}{\mathrm{d}y^4} - \frac{2H}{D_y}\left(\frac{p\pi}{a}\right)^2\frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} + Y\left[\frac{D_x}{D_y}\left(\frac{p\pi}{a}\right)^4 - \frac{\rho h\omega^2}{D_y}\right] = 0 \tag{25}$$

with boundary conditions, Eq. (7).

The characteristic equation corresponding to Eq. (25) is

$$\lambda^{*4} - \frac{2H}{D_y}\left(\frac{p\pi}{a}\right)^2\lambda^{*2} + \left[\frac{D_x}{D_y}\left(\frac{p\pi}{a}\right)^4 - \frac{\rho h\omega^2}{D_y}\right] = 0 \tag{26}$$

This characteristic equation has two imaginary and two real roots:

Real roots:

$$\lambda^*_{1,2} = \pm\lambda_1 = \pm\sqrt{\sqrt{\left(\frac{p\pi}{a}\right)^4\left[\left(\frac{H}{D_y}\right)^2 - \frac{D_x}{D_y}\right] + \frac{h\rho\omega^2}{D_y}} + \left(\frac{p\pi}{a}\right)^2\frac{H}{D_y}} \tag{27}$$

Imaginary roots:

$$\lambda^*_{3,4} = \pm i\lambda_3 = \pm i\sqrt{\sqrt{\left(\frac{p\pi}{a}\right)^4\left[\left(\frac{H}{D_y}\right)^2 - \frac{D_x}{D_y}\right] + \frac{h\rho\omega^2}{D_y}} - \left(\frac{p\pi}{a}\right)^2\frac{H}{D_y}} \tag{28}$$

The general solution can be represented as

$$Y(y) = A_1\cosh(y\lambda_1) + A_2\sinh(y\lambda_1) + A_3\cos(y\lambda_3) + A_4\sin(y\lambda_3) \tag{29}$$

In entirely the same way as in the MBM case, with Eq. (29) and with boundary conditions from Eq. (7) the following characteristic determinant can be derived:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \lambda_1 & 0 & \lambda_3 \\ C_2 & S_2 & c_2 & s_2 \\ \lambda_1 S_2 & \lambda_1 C_2 & -\lambda_3 s_2 & \lambda_3 c_2 \end{bmatrix} = 0 \qquad (30)$$

where in this case

$$c_2 = \cos(\lambda_3 a), \qquad s_2 = \sin(\lambda_3 a)$$
$$C_2 = \cosh(\lambda_1 a), \qquad S_2 = \sinh(\lambda_1 a) \qquad (31)$$

Expanding Eq. (30) and using Eqs. (27) and (28) gives

$$\sqrt{\frac{h\rho\omega^2}{D_y} - \frac{D_x}{D_y}\left(\frac{p\pi}{a}\right)^4}(1 - c_2C_2) + \frac{H}{D_y}\left(\frac{p\pi}{a}\right)^2 s_2S_2 = 0 \qquad (32)$$

Unlike the corresponding MBM, Eq. (14), Eq. (32) contains only one unknown quantity p. In the interval $m \leq p < m + 1$ there is only one root of Eq. (32). With the known root of Eq. (30), the mode shape in the y direction is

$$Y(y) = \cosh(\lambda_1 y) + \{1/[\lambda_1(1 - c_2C_2) - \lambda_3 s_2 S_2]\}$$
$$\times \{(\lambda_1/\lambda_3)[\lambda_3(1 - c_2C_2) + \lambda_1 s_2 S_2]\cos(\lambda_3 y)$$
$$+ (C_2\lambda_3 s_2 + c_2\lambda_1 S_2)[\sinh(\lambda_1 y) - (\lambda_1/\lambda_3)\sin(\lambda_3 y)]\} \qquad (33)$$

Second problem. The solution of Eq. (1) with ω from Eq. (24) is now taken as in Eq. (15). This solution has to satisfy the boundary conditions in Eq. (16). The solution of this problem is entirely analogous to the solution of the first problem. The following characteristic equation can be derived:

$$\sqrt{\frac{h\rho\omega^2}{D_x} - \frac{D_y}{D_x}\left(\frac{q\pi}{a}\right)^4}(1 - c_1C_1) + \frac{H}{D_x}\left(\frac{q\pi}{a}\right)^2 s_1S_1 = 0 \qquad (34)$$

where in this case

$$c_1 = \cos(\tilde{\lambda}_3 a), \qquad s_1 = \sin(\tilde{\lambda}_3 a)$$
$$C_1 = \cosh(\tilde{\lambda}_1 a), \qquad S_1 = \sinh(\tilde{\lambda}_1 a) \qquad (35)$$

$$\tilde{\lambda}_1 = \sqrt{\sqrt{\left(\frac{q\pi}{a}\right)^4\left[\left(\frac{H}{D_x}\right)^2 - \frac{D_y}{D_x}\right] + \frac{h\rho\omega^2}{D_x}} + \left(\frac{q\pi}{a}\right)^2\frac{H}{D_x}}$$
$$\tilde{\lambda}_3 = \sqrt{\sqrt{\left(\frac{q\pi}{a}\right)^4\left[\left(\frac{H}{D_x}\right)^2 - \frac{D_y}{D_x}\right] + \frac{h\rho\omega^2}{D_x}} - \left(\frac{q\pi}{a}\right)^2\frac{H}{D_x}} \qquad (36)$$

Equation (34) contains only one unknown quantity, q. In the interval $n \leq q < n + 1$, there exists only one root of Eq. (34). The mode shape in the x direction is then

$$X(x) = \cosh(\tilde{\lambda}_1 x) + \{1/[\tilde{\lambda}_1(1 - c_1C_1) - \tilde{\lambda}_3 s_1 S_1]\}$$
$$\times \{(\tilde{\lambda}_1/\tilde{\lambda}_3)[\tilde{\lambda}_3(1 - c_1C_1) + \tilde{\lambda}_1 s_1 S_1]\cos(\tilde{\lambda}_3 x)$$
$$+ (C_1\tilde{\lambda}_3 s_1 + c_1\tilde{\lambda}_1 S_1)[\sinh(\tilde{\lambda}_1 x) - (\tilde{\lambda}_1/\tilde{\lambda}_3)\sin(\tilde{\lambda}_3 x)]\} \qquad (37)$$

As already shown, for any mode w_{mn} the mode shape is determined as the product of the mode shape in the x direction and the mode shape in the y direction:

$$w_{mn} = X_{mn}(x) \cdot Y_{mn}(y) \qquad (38)$$

With the known mode shape the new frequency ω_{mn} corresponding to the mode is calculated by applying Rayleigh's method [see Eqs. (21–23)].

The formula equations (27–38) and (21–23) are repeated with this value of ω_{mn} and the recalculated values of p and q, and the new mode shape and corresponding frequency $\omega_{mn_{\text{new}}}$ are found. In the present calculation this process was repeated until the relative difference $|\omega_{mn_{\text{new}}} - \omega_{mn}|/\omega_{mn}$ was smaller than 10^{-9}. This degree of error was achieved in three to six iteration steps.

III. Numerical Results and Discussion

The special feature of the proposed approach is that it is much simpler than the MBM because it does not require the simultaneous solution of two transcendental equations with two variables. It yields a significant reduction of calculation time and renders the calculation process more stable. In Ref. 3 the importance of finding the region in which the roots of the transcendental equations are located before beginning the calculations was emphasized. After finding this region for any given mode, the MBM requires more than an hour of calculation to determine the frequency of this mode. By contrast, calculation of the frequency of any mode by the present approach takes no more than 5 min, less than 8% of the time required, to reach a solution by the MBM.

Numerical results for the problem in which all four edges of a square orthotropic plate are clamped are presented in Table 1. In this table the frequency parameter $\bar{\omega} = (a/\pi)^2\omega(\rho h/H)^{1/2}$ was obtained by 1) MBM, 2) Rayleigh's method (on the basis of mode shapes obtained from the MBM), and 3) the present approach. Table 1 reveals that the MBM yields the lowest values of the frequency parameter. It has been shown in the literature that although no mathematical proof exists BM yields lower bounds to the eigenvalues.[2,12] It is also seen in Table 1 that the percent differences between the MBM, Rayleigh's method, and the present approach were found to decrease with increasing value of the minimum(m, n).

Because of the asymptotic nature of the three approaches, the errors yielded by these methods are expected to be maximum for the least eigenvalue. It was, however, reported[12] that whereas the MBM estimate for the first eigenvalues is in error a few percent the error in the Rayleigh estimate is less than 0.17%. Table 1 shows that even for the least eigenvalue the difference between the Rayleigh estimation and that of the present approach is only 0.017%. This clearly indicates that values obtained by the present approach are closer to the true eigenvalues than are the MBM estimates. Hence, the accuracy of the present method of calculation, when linked with the substantial saving in computation time, shows that the method yields significant improvement over previous methods.

Numerical results for an isotropic plate clamped along one pair of opposite edges and free along the other pair are presented in Table 2 and for an orthotropic plate with similar boundary conditions in Table 3. In these tables numerical results obtained by the present solution are compared with results from Ref. 5, where they were calculated by the BM and by a series method.

An important facet of Bolotin-type solutions is that they permit the definition of vibration mode shapes as well as the natural frequencies, thus allowing for calculations of stresses and strains if required. This is especially true of the present approach by which mode shapes are defined during each step of iteration. Mode shapes obtained by the new approach for a plate clamped along all its edges are shown in Fig. 1.

As already pointed out (Tables 2 and 3), the BM failed to predict beam-like modes for a plate clamped along one pair of opposite edges and free along the other pair. This shortcoming, however, can be successfully overcome by application of the present modification. The present approach leads to the determination of mode shapes that are valid over a larger region of the plate than by the MBM. Because the MBM postulates the clamped-plate frequency in a form analogous to the corresponding frequency of a simply supported plate [Eq. (5)], it yields the mode shapes only beyond a certain distance from the edges of the plate. In contrast, the present approach is restricted only by the solutions of two Levy-type problems. The mode shapes thus obtained are valid everywhere on the plate, except near its corners. The advantage of the successful

Table 2 Natural frequencies of a square isotropic plate clamped along one pair of opposite edges and free along the other pair ($\nu = 0.3$)

	BM from Ref. 5			Present method			King and Lin,[5]
Number	p	q	$\omega a^2(\rho h/D)^{1/2}$	p	q	$\omega a^2(\rho h/D)^{1/2}$	$\omega a^2(\rho h/D)^{1/2}$
1	—	—	—	1.578	0.125	22.46	22.17
2	1.417	0.838	26.73	1.570	0.832	26.72	26.40
3	1.280	1.696	44.56	1.399	1.647	43.12	43.6
4	—	—	—	2.592	0.191	61.94	61.2
5	2.465	0.860	67.29	2.630	0.914	68.04	67.2
6	1.190	2.598	80.60	1.296	2.577	79.55	79.8
7	2.381	1.807	88.17	2.513	1.751	86.46	87.5

Table 3 Natural frequencies of a square orthotropic plate clamped along one pair of opposite edges and free along the other pair ($D_x/H = 1.543, D_y/H = 4.810, D_{xy}/H = 0.407$)

	BM from Ref. 5			Present method			King and Lin,[5]
Number	p	q	$\omega a^2(\rho h/H)^{1/2}$	p	q	$\omega a^2(\rho h/H)^{1/2}$	$\omega a^2(\rho h/H)^{1/2}$
1	——	——	——	1.522	0.056	27.78	27.73
2	1.450	0.772	32.79	1.525	0.639	30.81	31.63
3	1.342	1.591	66.19	1.431	1.542	63.15	64.56
4	——	——	——	2.521	0.168	76.44	76.40
5	2.477	0.864	82.50	2.542	0.749	80.78	81.77

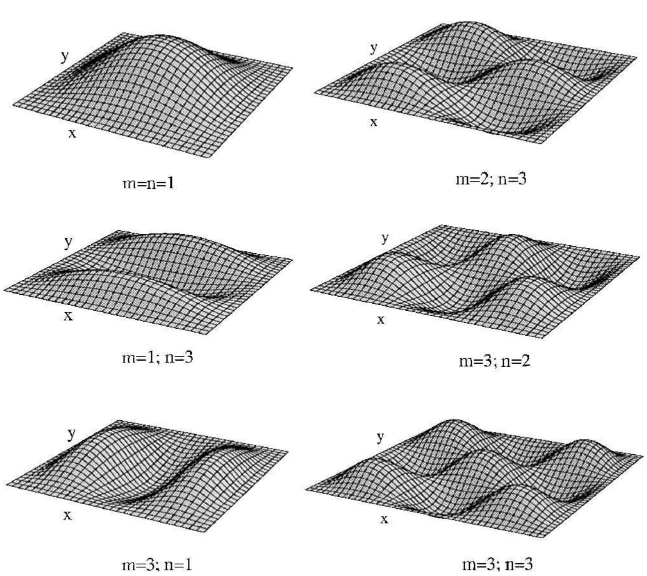

Fig. 1 Examples of the mode shapes of a plate clamped on all sides.

determination of the mode shapes over almost the entire plate area is obvious.

IV. Conclusions

A modification of the BM for predicting natural frequencies and mode shapes of thin rectangular plates was presented. The present approach, which is based on thin-plate theory, goes beyond the modification developed by Elishakoff[3] and Vijayakumar.[9] Calculated results for isotropic and orthotropic square plates were found to coincide with frequencies calculated by the Rayleigh method to within 0.017%. Furthermore, computations by the present modification are less laborious, yet more accurate than by the BM. From the initial calculation the frequency of a vibration, the next iteration, is derived in a straightforward process. From three to six iterations are sufficient to achieve a relative error less than 10^{-9}. Plates vibrating as beams with two free boundaries, although unsolvable by the BM, can easily be solved by the present approach. In contrast with the MBM, the present approach leads to mode shapes that are valid all over the plate, except for the corners of the plate.

Although the present approach is based on thin-plate theory, it is nevertheless equally applicable when considering thicker plates for which shear deformation theory is used.

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